Math 142 Lecture 23 Notes

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1 Knot Theory

1.1 Knot equivalence and isotopy

We have been using arguments with less rigor for two reasons.

1. The details are very similar to things we've already done the details for.
2. We have limited time to tackle as many interesting concepts as we can.

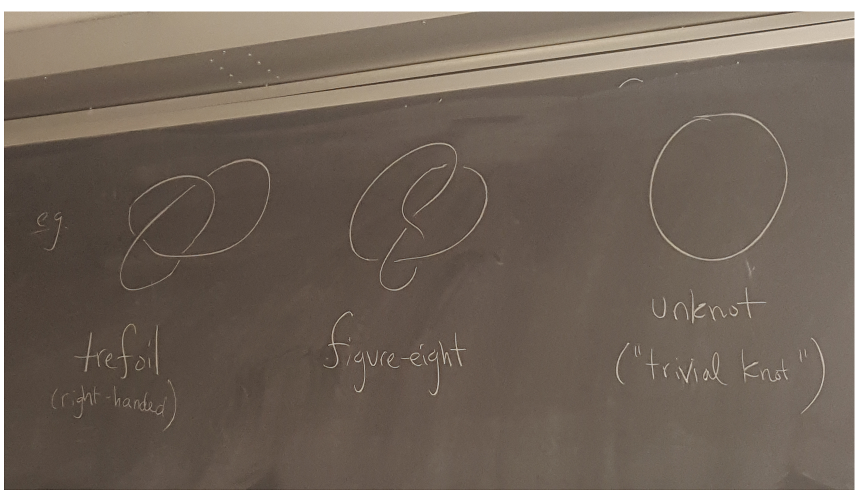
We may have to sacrifice rigor here, as well, for the same reasons.

Definition 1.1. A *knot* is a subspace of $\mathbb{R}^3$ that is homeomorphic to S^1.

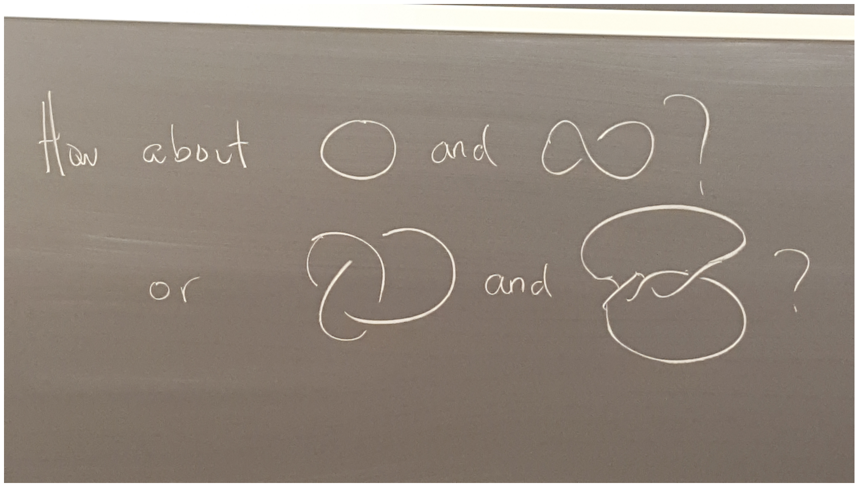
Definition 1.2. A *link* is a subspace of $\mathbb{R}^3$ that is homeomorphic to $S^1 \amalg \cdots \amalg S^1$.

We can't draw in 3D, so we need to draw projections of knots to the plane and keep track of over/under crossings.

Example 1.1.



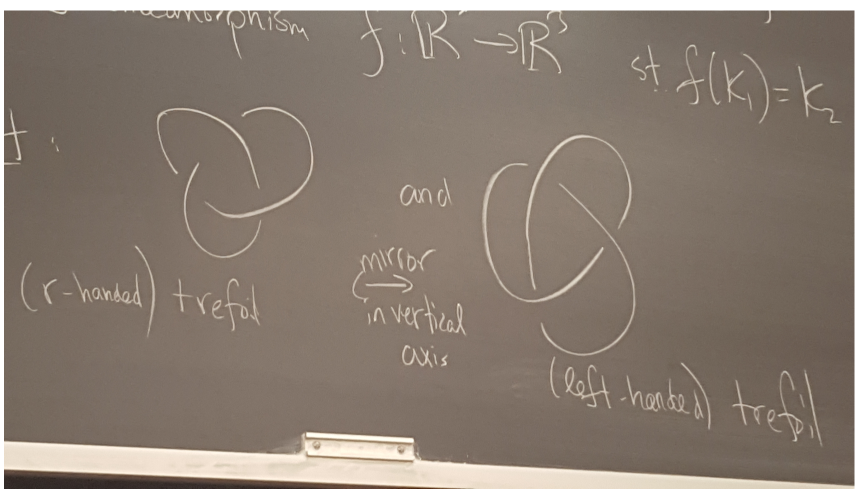
What about the difference between these knots?[1]



Definition 1.3. Two knots K_1 and K_2 are *equivalent* if there exists a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K_1) = K_2$.

This definition doesn't really get at the intuition people naturally have. When people think about knots, they think about rotating and bending knots and loops. However, here is another consideration.

Example 1.2. Take a right-handed trefoil and mirror it across a vertical axis to get a left-handed trefoil.

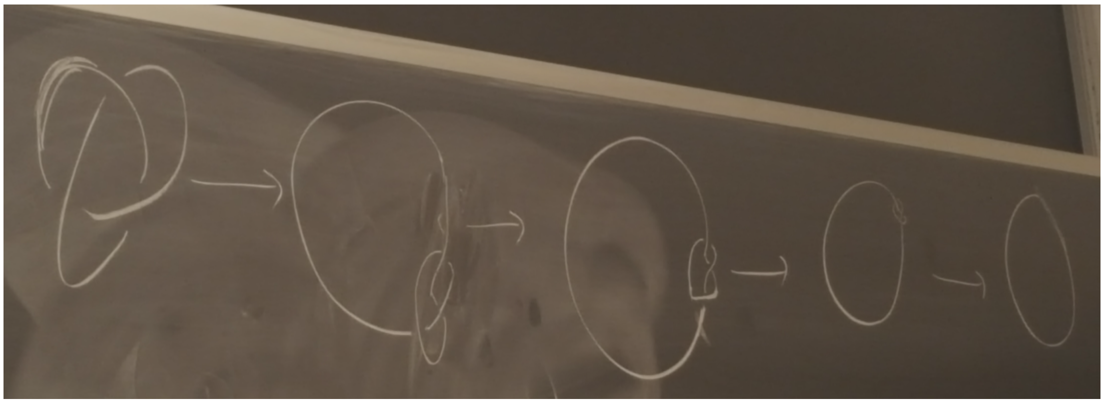


[1]Knot drawing is an acquired skill. There is a video online of William Thurston drawing the trefoil and figure-eight knot at the same time, one with each hand. The skill cap is high.

These trefoils are equivalent by reflection of $\mathbb{R}^3$. Although, one cannot "slide one onto the other" (not obvious).

What is a better notion of equivalence? We need to be careful:

Example 1.3. There exists a continuous family of knots interpolating between the trefoil and the unknot. Keep shrinking the tangled part of the knot.



This works for any knot, not just the trefoil.

Here is a better definition.

Definition 1.4. Two knots K_1 and K_2 are called *isotopic* if there exists a homotopy $F : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that

1. For every $t \in [0, 1]$, $F(x, t) : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism.
2. $F(x, 0) = \mathrm{id}_{\mathbb{R}^3}$
3. $F(K_1, 1) = K_2$.

Note that isotopic knots are equivalent.

Theorem 1.1. *K_1, K_2 are isotopic iff K_1, K_2 are equivalent via a homeomorphism f that is ambient isotopic to* $\mathrm{id}_{\mathbb{R}^3}$

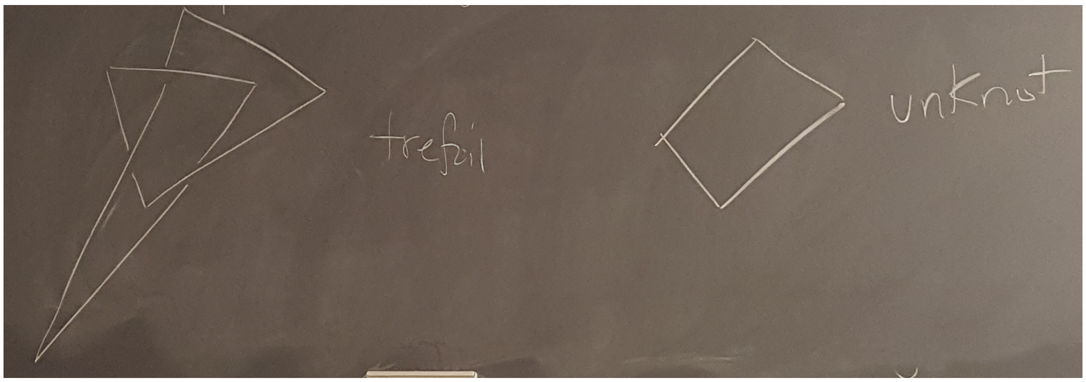
Example 1.4. The right-handed and left-handed trefoil knots are not isotopic.

1.2 Tame knots

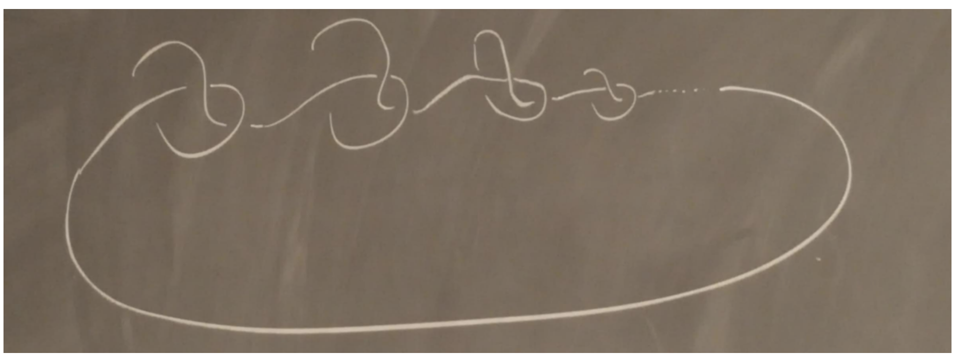
Knots, like topological spaces, can be very weird. We only care about knots that are equivalent to *polygonal knots*.

Definition 1.5. A *polygonal knot* is a knot that is comprised of finitely many line segments. Such knots are called *tame*; otherwise, the knot is called *wild*.

Example 1.5. Here are polygonal knots equivalent to the trefoil and the unknot, respectively.



Example 1.6. Here is an example of a wild knot.



From now on, we will only talk about tame knots. The following theorem says that we can draw tame knots, although we will not prove it.[2]

Theorem 1.2. *Every (tame) knot is isotopic to one whose projection to the* (x, y)*-plane is nice, i.e. with finitely many double points (two line segments intersecting), no triple (or more) points, and no tangencies.*

If you have studied any differential topology, this is saying that we want intersections to be transverse to each other.

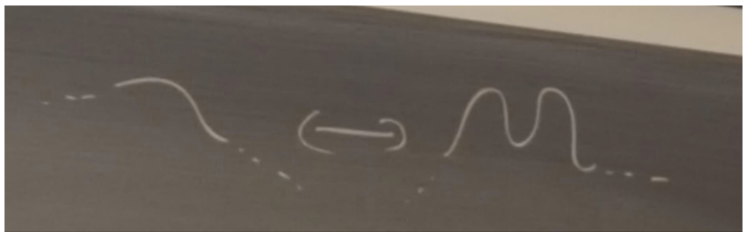
[2]Professor Conway says we won't prove this because it's not fun, and we're having fun right now.

1.3 Reidemeister moves

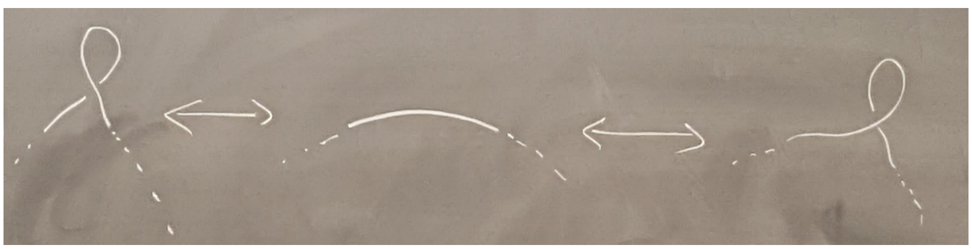
There are "moves" on a projection that don't change the isotopy class of the corresponding knot.

Definition 1.6. The *Reidemeister moves* are the following transformations of a projection of a knot:

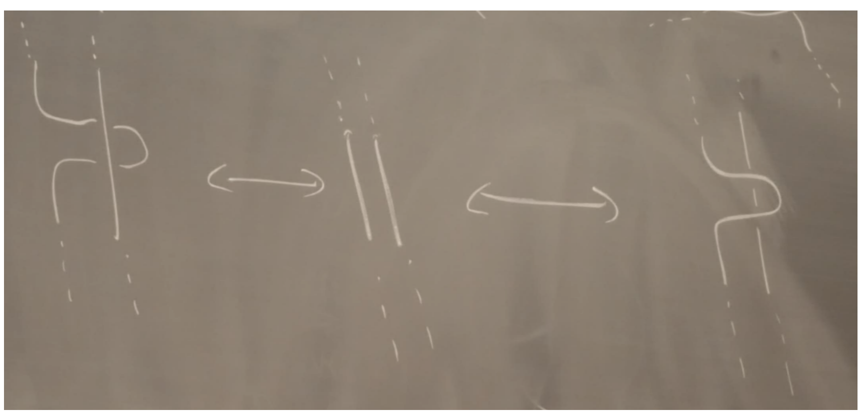
- (R0) We can use any "isotopy in the plane" (homeomorphism of $\mathbb{R}^2$ that is homotopic to $\mathrm{id}_{\mathbb{R}^2}$)



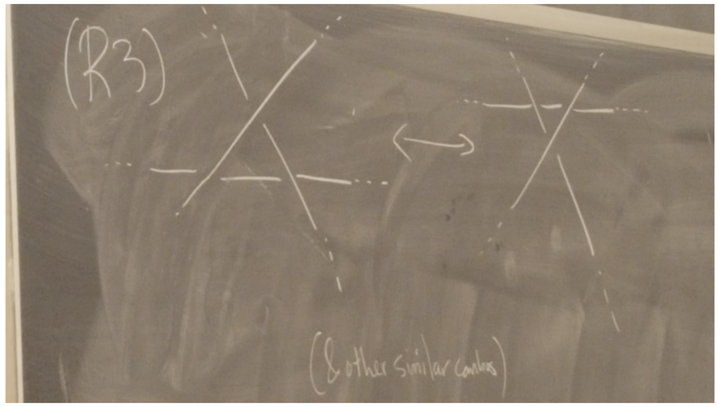
- (R1) We can loop or unloop part of a knot.



- (R2) We can move part of a knot under another part.



- (R3) We can slide part of a knot around if it is below or above a crossing of two other parts of the projection.



Theorem 1.3 (Reidemeister (1927),Alexander-Baird-Briggs (1926)[3]). *If two nice projections are of isotopic knots, then the projections are related by a finite sequence of Reidemeister moves (and lots of R(0)).*

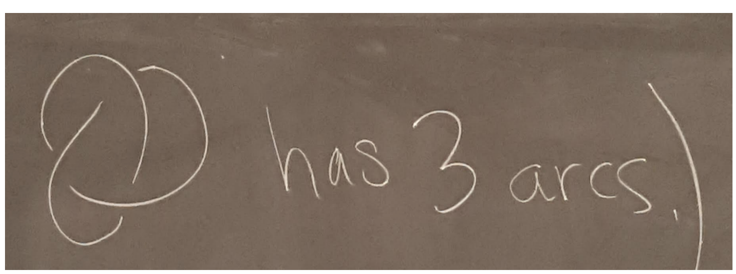
How many moves does it take? We know that it is $\leq 2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$, where n is the number of crossings X in both diagrams and the number of $2s$ is $10^{1000000n}$.[4]

How do we understand knots from their projections, then? We want to define invariants of isotopy classes of knots by defining invariants of nice projections that don't change when doing R1-R3.

1.4 Tricolorability

The idea is to color the arcs of a nice projection in a certain way. We will count the number of ways we can do such a coloring of a given knot. We will explicitly define what an arc is next time.

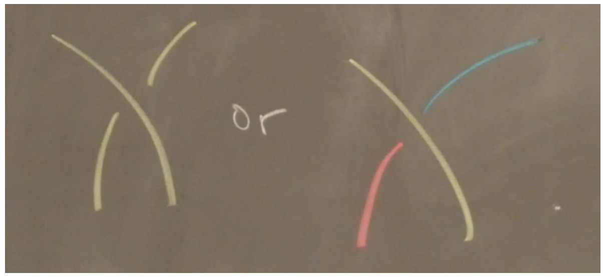
Example 1.7. The trefoil knot has 3 arcs, indicated by how we draw it.



[3]The proofs were done independently.

[4]This is the most current bound, as of 2014.

Definition 1.7. A *tricoloring* of a projection is a coloring of the arcs of the projection (red/green/blue or 1/2/3) such that at each crossing, either each color is the same, or all are different.



Definition 1.8. A *trivial coloring* is one that uses one color.

Definition 1.9. A knot is *tricolorable* if there exists a (nice) projection that has a non-trivial tricoloring.

Example 1.8. The trefoil knot is tricolorable.